Absolute (Global) Extreme Values

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DEFINITION **Absolute Extreme Values**

Let f be a function with domain D. Then $f(c)$ is the

(a) **absolute maximum value** on D if and only if $f(x) \leq f(c)$ for all x in D.

(b) **absolute minimum value** on D if and only if $f(x) \geq f(c)$ for all x in D.

EXAMPLE 1 **Exploring Extreme Values**

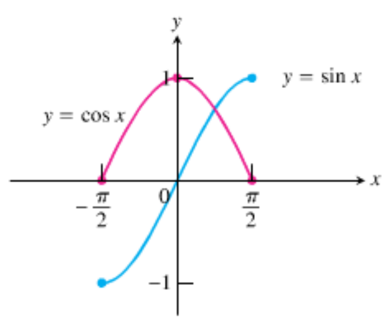
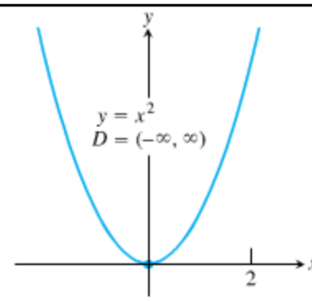
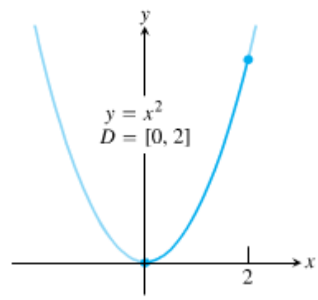


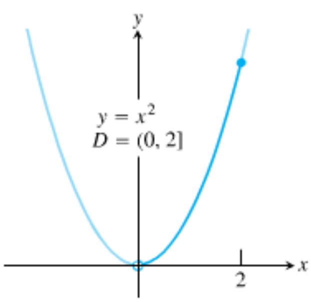
Figure 4.1 (Example 1)

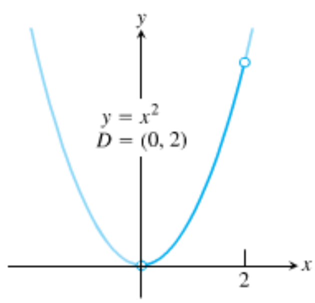
EXAMPLE 2 Exploring Absolute Extrema



	Function Rule	Domain D	Absolute Extrema on D
(a)	$y = x^2$	$(-\infty, \infty)$	
(b)	$y = x^2$	$[0, 2]$	
(c)	$y = x^2$	$(0, 2]$	
(d)	$y = x^2$	$(0, 2)$	







Example 2 shows that a function may fail to have a maximum or minimum value. This cannot happen with a continuous function on a finite closed interval.

THEOREM 1 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f has both a maximum value and a minimum value on the interval. (Figure 4.3)

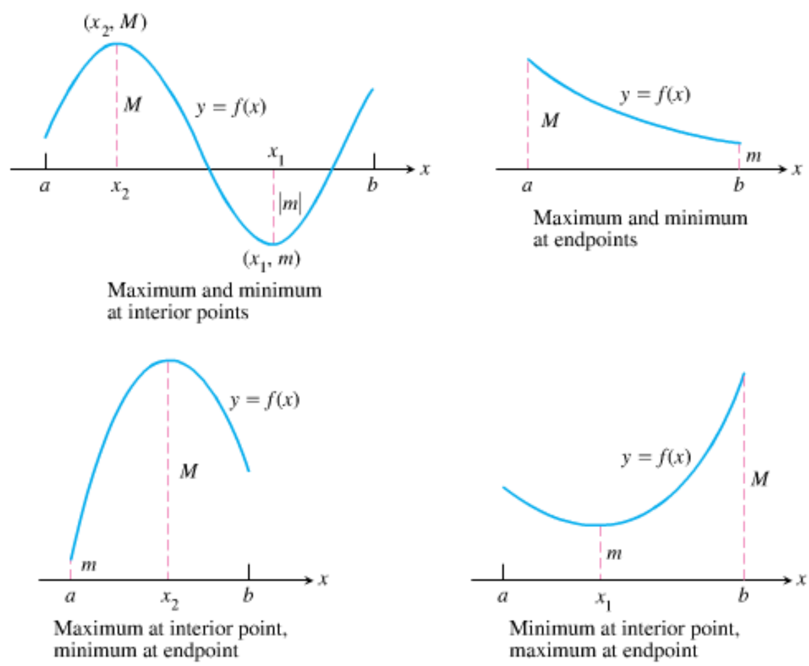


Figure 4.3 Some possibilities for a continuous function's maximum (M) and minimum (m) on a closed interval $[a, b]$.

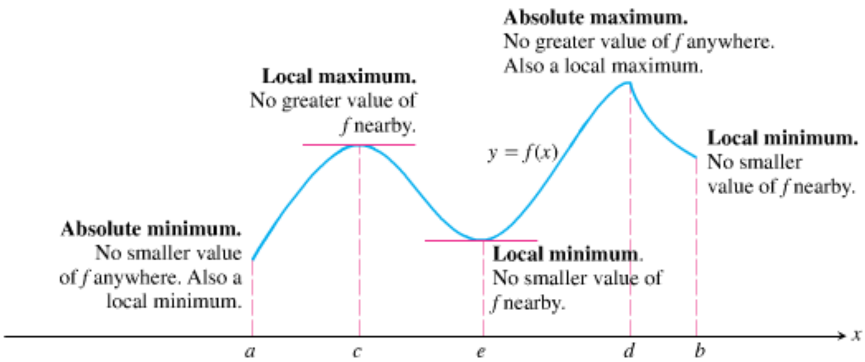


Figure 4.4 Classifying extreme values.

Local extrema are also called **relative extrema.**

DEFINITION Local Extreme Values

Let c be an interior point of the domain of the function f. Then $f(c)$ is a

(a) **local maximum value** at c if and only if $f(x) \leq f(c)$ for all x in some open interval containing c.

(b) **local minimum value** at c if and only if $f(x) \geq f(c)$ for all x in some open interval containing c.

A function f has a local maximum or local minimum *at an endpoint* c if the appropriate inequality holds for all x in some half-open domain interval containing c.

THEOREM 2 Local Extreme Values

If a function f has a local maximum value or a local minimum value at an interior point c of its domain, and if f' exists at c, then

$$f'(c) = 0.$$

An **absolute extremum** is also a local extremum.

DEFINITION Critical Point

A point in the interior of the domain of a function f at which $f' = 0$ or f' does not exist is a **critical point** of f.

DEFINITION Stationary Point

A point in the interior of the domain of a function f at which $f' = 0$ is called a ***stationary point*** of f.

EXAMPLE 3 Finding Absolute Extrema

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

EXAMPLE 4 Finding Extreme Values

Find the extreme values of $f(x) = \dfrac{1}{\sqrt{4 - x^2}}$.

While a function's extrema can occur only at critical points and endpoints, not every critical point or endpoint signals the presence of an extreme value.

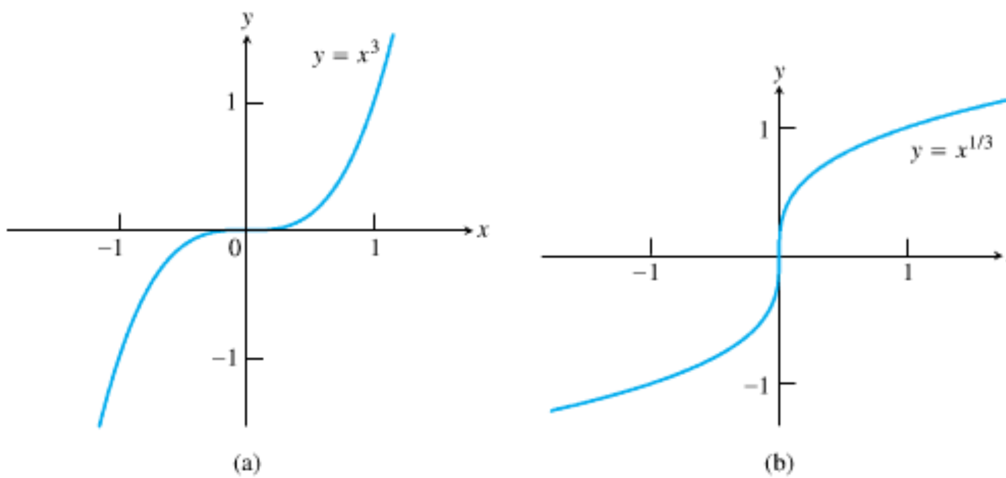


Figure 4.7 Critical points without extreme values. (a) $y' = 3x^2$ is 0 at $x = 0$, but $y = x^3$ has no extremum there. (b) $y' = (1/3)x^{-2/3}$ is undefined at $x = 0$, but $y = x^{1/3}$ has no extremum there.

EXAMPLE 5 Finding Extreme Values

Find the extreme values of

$$f(x) = \begin{cases} 5 - 2x^2, & x \le 1 \\ x + 2, & x > 1. \end{cases}$$

domain: $(-\infty, \infty)$

$$f'(x) = \begin{cases} -4x & x < 1 \\ 1 & x > 1 \end{cases}$$

$f'(1)$ does not exist because $\lim_{x \to 1^-} f'(x) = -4$

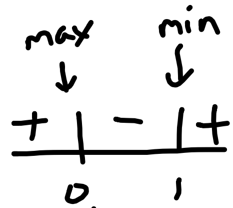
$\lim_{x \to 1^+} f'(x) = 1$

Critical point at $x = 1$

$-4x = 0$

$x = 0$

Critical point at $x = 0$



local max at $x = 0$ because

$f'(x) > 0$ when $x < 0$

$f'(x) < 0$ when $0 < x < 1$

$f(0) = 5 - 2(0)^2 = 5$

local min at $x = 1$ because

$f'(x) < 0$ when $0 < x < 1$

$f'(x) > 0$ when $x > 1$

$f(1) = 5 - 2(1)^2 = 3$

EXAMPLE 6 Using Graphical Methods

Find the extreme values of $f(x) = \ln\left|\frac{x}{1+x^2}\right|$.

SOLUTION

domain: $(-\infty, 0) \cup (0, \infty)$

$$f'(x) = \frac{1+x^2}{x} \cdot \frac{1(1+x^2) - 2x(x)}{(1+x^2)^2}$$

$$= \frac{\cancel{(1+x^2)}(1-x^2)}{x(1+x^2)^2} = \frac{(1-x^2)}{x(1+x^2)}$$

critical pts

$f'(x) = 0$ at $x = 1$, $x = -1$

$f'(x)$ is undefined at $x=0$ undefined ($x=0$ not in the domain)

max max

$+ \mid - \mid + \mid -$

$-1 \quad 0 \quad 1$

lots of words for explanation

$f(-1) = \ln\left|-\frac{1}{2}\right|$

$= \ln\frac{1}{2}$

$f(1) = \ln\left|\frac{1}{2}\right|$

$= \ln\frac{1}{2}$

absolute max of $\ln\frac{1}{2}$ at $x=-1$ and $x=1$